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A wave-function for stringy universes*

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Abstract: We define a wave-function for string theory cosmological backgrounds. We give a prescription for computing its norm following an earlier analysis within general relativity. Under Euclidean continuation, the cosmologies we discuss in this paper are described in terms of compact parafermionic worldsheet systems. To define the wave-function we provide a T-fold description of the parafermionic conformal field theory, and of the corresponding string cosmology. In specific examples, we compute the norm of the wave-function and comment on its behavior as a function of moduli.

Keywords: dS vacua in string theory, Conformal Field Models in String Theory.

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Contents

1. Introduction

Our goal in this paper is to embed the Hartle-Hawking no-boundary proposal for a wave-function description of the quantum state of the universe [1, 2] in a string theoretic framework. The Hartle-Hawking proposal pertains in particular to de Sitter-like universes in general relativity. A partial list of interesting recent work on related topics is [3 − 8].

One motivation for embedding the Hartle-Hawking proposal into string theory is that it provides us with a calculable quantity in de Sitter-like compactifications of a quantum theory of gravity. These quantities are hard to come by (see e.g. [9, 10] and references thereto).

Two essential properties which a cosmological background must fulfill in order to admit a wave-function description under the no-boundary proposal are the following. First the cosmology must be spatially closed. More importantly, the cosmology should admit a continuation to a positive definite Euclidean geometry that is compact and has no boundaries or singularities. The most familiar example is the case of n-dimensional de Sitter space, dS_n, where these properties are satisfied. In global coordinates, the dS_n metric is given by

$$ds^2 = R^2(-dt^2 + \cosh^2 t\; d\Omega^2), \tag{1.1}$$

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where $d\Omega^2$ is the metric on a round unit $(n-1)$-sphere and R is the radius of curvature. The spatial slices of constant time t are $(n-1)$-spheres of radius $R\cosh t$. We can rotate to Euclidean signature by setting $t = i\tau = i(\pi/2 - \theta)$, upon which we obtain an n-sphere S_n of radius R. The Euclidean continuation is a compact smooth manifold.

In a field theoretic framework, the quantum state of a de Sitter cosmology can be expressed as a functional of fields, including both matter fields and metric fluctuations, on a spatial slice of time-reversal symmetry. For the reversal $t \to -t$, this is the slice $t = 0$ in the de Sitter space dS_n. That slice is also the equator $\theta = \pi/2$ of the corresponding Euclidean sphere S_n. Imagine cutting de Sitter space along this slice and gluing smoothly one half of it to half a sphere S_n. Under the Hartle-Hawking proposal, we express the wave-function as a Euclidean path integral over half the sphere S_n with the condition that the metric g_{ij} and the matter fields, collectively denoted by ϕ, take specific values (h_{ij}, ϕ_0) on the boundary equator $\theta = \pi/2$:

$$\Psi(h_{ij}, \phi_0) = \int [dg][d\phi] e^{-S_E(g,\phi)}. \tag{1.2}$$

No other boundary condition needs to be specified due to the compactness of the Euclidean manifold. Here, S_E is the Euclidean gravitational action in the presence of matter fields and a positive cosmological constant Λ.

The norm of the wave-function is given by the full Euclidean path integral on S_n. It can be computed in the semi-classical approximation by evaluating the Euclidean action for a given solution to the classical equations of motion. One solution is empty de Sitter space of radius $R \sim \Lambda^{-1/2}$. In this approximation, and in the case of four dimensions, the norm is given by [1]:

$$||\Psi_{HH}||^2 \sim e^{\frac{2}{3\lambda}} , \tag{1.3}$$

where the dimensionless parameter λ is proportional to the cosmological constant:

$$\lambda = 2G\Lambda/9\pi \ .$$

The compactness of the Euclidean manifold ensures that the full path integral is free of any infrared divergences. However the field theory in question is non-renormalizable, and to go beyond the semi-classical approximation, we need to impose an ultraviolet cutoff. One way to deal with the ultraviolet ambiguities is to embed the calculation in a string theoretic framework, where we expect the ultraviolet divergences to be absent. Unfortunately there are no known classical de Sitter solutions in string theory to begin with. Therefore, we seek other cosmological backgrounds which are exact solutions to string theory and for which we can generalize the Hartle-Hawking computation.

To this end, notice that any tachyon free, compact Euclidean string background provides us with a finite, calculable quantity, namely the string partition function Z_{string}. Associated to the classical string background is a two-dimensional worldsheet conformal field theory (CFT). At the perturbative level, the string partition function can be computed as usual as a sum of CFT vacuum amplitudes over compact worldsheets of all topologies. Our proposal is that when such Euclidean string backgrounds admit a continuation to a

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Lorentzian cosmology, the Hartle-Hawking construction can be generalized with the norm of the wave-function given by

$$||\Psi_{cosm.}||^2 = e^{Z_{\text{string}}}. \tag{1.4}$$

We will motivate this formula by working out specific examples in string perturbation theory. As we will explain, the relevant string partition function has to be thermal.

Given the discussion above, a first candidate to consider is a Euclidean model for which the two-dimensional CFT is of the form $\mathrm{SU}(2)_k \times K$, the first factor corresponding to an SU(2) Wess-Zumino-Witten (WZW) model at level k and the second factor K corresponding to a suitable internal compact conformal field theory. The WZW factor is equivalent to a sigma model on a 3-sphere of radius $(k\alpha')^{1/2}$ and with k units of NSNS 3-form flux through the sphere. The dilaton field is constant and by choosing this to be small we can apply string perturbation theory. Unfortunately, however, the continuation to Lorentzian signature results in a dS_3 cosmology with imaginary flux, and it is not clear whether such a Lorentzian background is physical. (See [11] for an alternative non-compact, time-like Liouville model for which the SU(2) WZW factor describes the internal space.)

The only known string theory example which satisfies all the criteria we described so far is based on the parafermionic $\mathrm{SU}(2)_{|k|}/\,\mathrm{U}(1)$ coset model [12], which can be realized as a gauged SU(2) WZW model at level $|k|$. We consider Euclidean backgrounds corresponding to a two-dimensional CFT of the form $\mathrm{SU}(2)_{|k|}/\,\mathrm{U}(1) \;\times\; K$ where K is again an internal compact conformal field theory. Such a Euclidean background admits a Lorentzian continuation to a cosmological background belonging to a class of models studied in [13, 14], and which are described by two-dimensional CFTs of the form $\mathrm{SL}(2,R)_{-|k|}/\,\mathrm{U}(1) \;\times\; K$. To avoid having to deal with the tachyonic instabilities of bosonic string theory, we consider solutions of this form in superstring theory. The total central charge must be $c_{\text{tot}} = 15$ ($\hat{c} = 10$) in order for worldsheet (super-)gravitational anomalies to cancel. When we fix the internal conformal field theory K, the level $|k|$ is determined by anomaly cancellation.

The non-trivial time-dependence of the cosmology necessarily breaks space-time supersymmetry. As in the de Sitter case, the Euclidean path integral can be interpreted as a thermal ensemble. Thus from the point of view of the Euclidean $N = 2$ worldsheet superconformal system, space-time supersymmetry will be broken by specific boundary conditions, analogous to the thermal co-cycles that appear in the partition function of superstring theories on flat space at finite temperature [15]. For large level $|k|$, the effective temperature of the models is of order $T \sim 1/\sqrt{|k|\alpha'}$ [14, 27]. In this paper, we will explore some low level $|k|$ models. In order for the corresponding cosmological wave-function to be computable in string perturbation theory, the effective temperature must be below the Hagedorn temperature. A Hagedorn temperature would signal a phase transition, as proposed in [15 − 17]. We will construct explicitly low level $|k|$ models for which the effective temperature is below the Hagedorn temperature and so string perturbation theory can be applied.

It is well known that the geometric sigma model approach to the parafermionic coset model (and to the corresponding Lorentzian cosmology) leads to a metric with curvature singularities and strong coupling. However, the underlying CFT is perfectly well behaved at these apparently singular regions, and by using T-duality a weakly coupled description of these regions can be obtained [18]. Using this fact, we construct an almost geometrical description of the CFT in terms of a compact, non-singular T-fold [19, 20] with a well-defined partition function. These considerations allow us to define the wave-function of the Lorentzian cosmology.

Our paper is organized as follows. In section 2, we review properties of the two-dimensional $\mathrm{SL}(2,R)_{-|k|}/\mathrm{U}(1) \times K$ conformal field theory that corresponds to a cosmological background. It is the analogue of the de Sitter universe. In section 3, we describe how to analytically continue the cosmology to a compact Euclidean space-time described at the string level by a two-dimensional parafermionic model of the form $\mathrm{SU}(2)_{|k|}/\mathrm{U}(1) \times K$. Then, we discuss in section 4 how to obtain an almost geometrical description of these backgrounds in terms of T-folds. We discuss in sections 5 and 6 how to calculate a wave-function and its norm for the cosmology. In section 7 we discuss the thermal nature of the wave-function. In section 8 we apply the definition of the wave-function to some particular compact models and for which perturbation theory can be used to compute its norm. Finally we discuss interpretations of the results in the concluding sections.

2. The cosmological solution

In this section, we review in some detail the cosmological solution of string theory which is based on an $\mathrm{SL}(2,R)/\mathrm{U}(1)$ gauged WZW model at level k [13].

We can define a WZW conformal field theory on the group manifold $\mathrm{SL}(2,R)$, at least classically. The worldsheet action is given by

$$S = \frac{k}{4\pi}\int_\Sigma d^2z Tr(g^{-1}\partial g g^{-1}\bar{\partial} g) + \frac{ik}{12}\int_M Tr(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg), \qquad (2.1)$$

where Σ is the worldsheet Riemann surface, M is a 3-manifold whose boundary is Σ and g is an element of $\mathrm{SL}(2,R)$.

For concreteness, we parameterize the $\mathrm{SL}(2,R)$ group manifold as follows

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} \qquad (2.2)$$

with $ab + uv = 1$. The conformal field theory has an $\mathrm{SL}(2,R) \times \mathrm{SL}(2,R)$ global symmetry. We choose to gauge an axial $\mathrm{U}(1)$ subgroup under which $g \to hgh$. In particular, we consider the non-compact $\mathrm{U}(1)$ subgroup generated by

$$\delta g = \epsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g + \epsilon g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (2.3)$$

Infinitesimally, we have the transformations $\delta a = 2\epsilon a$, $\delta b = -2\epsilon b$, $\delta u = \delta v = 0$. To gauge this $\mathrm{U}(1)$ symmetry, we introduce an Abelian gauge field and render the action invariant. The action is quadratic and non-derivative in the gauge field, and so this can be integrated out in a straightforward way [21].

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In the region $1 - uv > 0$, we can use the gauge freedom to set $a = b$ and integrate out the gauge field. The resulting action is expressed in terms of gauge invariant degrees of freedom only, and it turns out to be

$$S = -\frac{k}{4\pi} \int d^2 z \frac{\partial u \bar{\partial} v + \partial v \bar{\partial} u}{1 - uv}, \tag{2.4}$$

while a non-trivial coupling to the worldsheet curvature is generated corresponding to a dilaton background [21]. This action can be identified with a non-linear sigma-model action with background metric

$$ds^2 = -k\alpha' \frac{dudv}{1 - uv}. \tag{2.5}$$

The non-trivial dilaton is given by

$$e^{2\Phi} = \frac{e^{2\Phi_0}}{1 - uv}. \tag{2.6}$$

The metric (2.5) is a Lorentzian metric whose precise causal structure, however, depends on the sign of k. For positive level k, u and v are Kruskal-like null coordinates of a 2-dimensional black hole. In this case, the time-like coordinate is given by $u + v$, and the metric has space-like singularities in future and past times at $uv = 1$.

For negative level k, one obtains a cosmological solution [13]. It consists of a singularity-free light-cone region, and there are (apparent) time-like singularities in the regions outside the light-cone horizons. Indeed, for negative level k we may set $u = -T + X$ and $v = T + X$ and the metric becomes

$$ds^2 = |k|\alpha' \frac{-dT^2 + dX^2}{1 + T^2 - X^2}. \tag{2.7}$$

The surfaces of constant time T intersect the singularities at $X = \pm\sqrt{1 + T^2}$. Even though the singularities follow accelerated trajectories, their proper distance remains finite with respect to the string frame metric

$$L = (|k|\alpha')^{\frac{1}{2}} \int_{-\sqrt{1+T^2}}^{\sqrt{1+T^2}} \frac{dX}{\sqrt{1 + T^2 - X^2}} = \pi(|k|\alpha')^{\frac{1}{2}}. \tag{2.8}$$

So with respect to stringy probes, the cosmology is spatially closed.

The singularity-free light-cone region is the region $T^2 - X^2 \geq 0$ (or $uv \leq 0$). The future part of this region describes an expanding, asymptotically flat geometry with the string coupling vanishing at late times. See e.g. [11, 13, 14, 23−28] for some discussions of these types of models. To see this, we parameterize the region $uv \leq 0$ with coordinates (x, t) such that

$$u = -te^{x}, \quad v = te^{-x} \tag{2.9}$$

and the metric becomes

$$ds^2 = |k|\alpha' \frac{-dt^2 + t^2 dx^2}{1 + t^2}, \tag{2.10}$$

while the dilaton field becomes

$$e^{2\Phi} = \frac{e^{2\Phi_0}}{1 + t^2}. \tag{2.11}$$

The scalar curvature is given by

$$\mathcal{R} \sim \frac{1}{|k|\alpha'(1+t^2)}. \tag{2.12}$$

Initially the curvature is set by the level $|k|$ and it is positive. No matter how small the level $|k|$ is, asymptotically the scalar curvature vanishes. An observer in this region never encounters the singularities. These are hidden behind the visible horizons at $T = \pm X$. However signals from the singularities can propagate into the region $uv < 0$, and therefore influence its future evolution.

Thus when $|k|$ is small the early universe region $t \sim 0$ is highly curved, with curvature of order the string scale. In this sense, it is similar to a big-bang cosmology. Despite the regions of large curvature, this cosmological background has a well defined CFT description and can be described in a string theoretic framework.

The cosmological background can also be realized as a solution of superstring theory by generalizing the worldsheet theory to a superconformal $\mathrm{SL}(2,R)/\,\mathrm{U}(1)$ model. The central charge of the superconformal $\mathrm{SL}(2,R)/\,\mathrm{U}(1)$ model at negative level k, is given by

$$c = 3 - \frac{6}{|k|+2}, \quad \hat{c} = 2 - \frac{4}{|k|+2} \tag{2.13}$$

In superstring theory, we must tensor it with other conformal field theories so as to satisfy the condition $\hat{c}_{\text{tot}} = 10$ for worldsheet gravitational anomalies to cancel.

An interesting case considered in [13] is the case where we add two large (however compact) free super-coordinates (y, z) together with a compact, superconformal CFT of central charge $\delta\hat{c} = 6 + 4/(|k|+2)$. The resulting background is a four dimensional cosmological background whose metric in Einstein frame is given by

$$ds_E^2 = |k|\alpha'(-dt^2 + t^2 dx^2) + (1+t^2)(R_y^2 dy^2 + R_z^2 dz^2). \tag{2.14}$$

This is an anisotropic cosmology which at late times however, and for large $R_y \sim R_z$, asymptotes to an isotropic flat Friedman-Robertson cosmology.

The cosmological region $t^2 = -uv \geq 0$ is non-compact, and when $R_{y,z}$ are large it has the desired four-dimensional interpretation. This is so *irrespective of how small the level k is.* In the region $uv > 0$ ($t^2 < 0$), sigma-model time-like singularities appear at $uv = 1$ ($t^2 = -1$). As we propose later in this work, these singularities are resolved at the string level, since the structure of the space-time manifold is replaced by a non-singular T-fold.

The string partition function depends crucially on the extra $\hat{c} = 6 + 4/(|k|+2)$ superconformal system, which is taken to be compact. In contrast to the four dimensional part defined by (t, x, y, z), for the internal, Euclidean $\hat{c} = 6 + 4/(|k|+2)$ system, the naive six-dimensional interpretation, which is valid for large level $|k|$ with curvature corrections of order $1/(|k|\alpha')$, is not valid for small values of $|k|$ [18]. Indeed for example for $|k| = 2$, the system can be taken to be a *seven*-dimensional torus. In general, small $|k|$ implies that the generalized curvatures (i.e. including dilaton gradients etcetera) are large and the moduli/radii are small. We remind the reader of the example of the $\mathrm{SU}(2)_{k=1}$ Wess-Zumino-Witten model which is equivalent to a (one-dimensional) compact boson at self dual radius. For large level $|k|$, however, the sigma model manifold is a large three-dimensional sphere with NSNS 3-form flux.

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3. The Euclidean continuation

Let us consider the region $1-uv \geq 0$ of the two-dimensional cosmology, and set $u = -T + X, v = T + X$. We can rotate to Euclidean signature by setting $T \to -iT_E$. The Euclidean continuation is a disk of unit coordinate radius parameterized by $Z = X + iT_E, \bar{Z} = X - iT_E$ such that $|Z|^2 \leq 1$. The metric (2.5) becomes

$$ds^2 = |k|\alpha' \frac{dZd\bar{Z}}{1 - Z\bar{Z}} = |k|\alpha' \frac{d\rho^2 + \rho^2 d\phi^2}{1-\rho^2} \tag{3.1}$$

and the dilaton

$$e^{2\Phi} = \frac{e^{2\Phi_0}}{1 - Z\bar{Z}} = \frac{e^{2\Phi_0}}{1-\rho^2}, \tag{3.2}$$

where we have also set $Z = \rho e^{i\phi}$ with $0 \leq \rho \leq 1$. The singularity becomes the boundary circle $\rho = 1$.

The radial distance of the center to the boundary of the disk is finite, but the circumference of the boundary circle at $\rho = 1$ is infinite. Geometrically the space looks like a bell. This Euclidean background corresponds to a well defined worldsheet conformal field theory based on an SU(2)/ U(1) gauged WZW model at level $|k|$.

From the point of view of the WZW worldsheet theory, the Euclidean continuation can be understood as a double analytic continuation as follows. We parameterize the SL$(2, R)$ group manifold as in equation (2.2). Let us also set $a = \tilde{X} - \tilde{T}, b = \tilde{X} + \tilde{T}$ so that the group element becomes

$$g = \begin{pmatrix} \tilde{X} - \tilde{T} & X - T \\ -X - T & \tilde{X} + \tilde{T} \end{pmatrix} \tag{3.3}$$

with

$$\tilde{X}^2 + X^2 - \tilde{T}^2 - T^2 = 1. \tag{3.4}$$

This parameterization shows that the SL$(2, R)$ group manifold is a 3-dimensional hyperboloid. Then it is clear that upon the double analytic continuation $T \to -iT_E, \tilde{T} \to -i\tilde{T}_E$ the group element becomes the following SU(2) matrix

$$g = \begin{pmatrix} W & Z \\ -\bar{Z} & \bar{W} \end{pmatrix} \tag{3.5}$$

with $W\bar{W} + Z\bar{Z} = 1$. After the analytic continuation we also have that $a \to W = \tilde{X} + i\tilde{T}_E, b \to \bar{W} = \tilde{X} - i\tilde{T}_E$.

A useful parameterization of the SU(2) group manifold for our purposes is

$$W = \cos\theta e^{i\chi}, \;\; Z = \sin\theta e^{i\phi} \tag{3.6}$$

and the metric on S^3 in these coordinates becomes

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\chi^2. \tag{3.7}$$

The ranges of the angles are as follows $0 \leq \theta \leq \pi, 0 \leq \chi, \phi, \leq 2\pi$.

The original global SL$(2, R) \times$ SL$(2, R)$ symmetry naturally continues to the SU$(2) \times$ SU(2) global symmetry of the resulting SU(2) WZW model. The non-compact U(1) axial symmetry subgroup that we gauge continues to a compact U(1) subgroup generated by

$$\delta g = i\epsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g + i\epsilon g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.8}$$

which amounts to the following infinitesimal transformations $\delta W = 2i\epsilon W$, $\delta\bar{W} = -2i\epsilon\bar{W}$ and $\delta Z = \delta\bar{Z} = 0$. In the parameterization (3.6), the U(1) symmetry corresponds to shifts of the angle χ. Gauging this symmetry results in the SU(2)/ U(1) coset model. In the Euclidean set-up, we take the level $|k|$ to be an integer for the WZW model to be well-defined.

After the analytic continuation described, we end up with the action (see e.g. [30] for a review):

$$S = \frac{|k|}{2\pi}\int d^2z\partial\theta\bar{\partial}\theta + \tan^2\theta\partial\phi\bar{\partial}\phi + \cos^2\theta(\partial\chi + \tan^2\theta\partial\phi + A_z)(\bar{\partial}\chi - \tan^2\theta\bar{\partial}\phi + A_{\bar{z}}). \tag{3.9}$$

In the Euclidean theory the gauge freedom can be fixed by setting the imaginary part of W (equivalently the angle χ) to zero. The equations of motion for the gauge field can then be used to integrate the gauge field out. This amounts to setting the last term in (3.9) to zero and producing a dilaton $e^{2\Phi} = e^{2\Phi_0}/\cos^2\theta$. We end up with a sigma model action with metric

$$ds^2 = |k|\alpha'(d\theta^2 + \tan^2\theta d\phi^2) \tag{3.10}$$

which is equivalent to the metric (3.1) after the coordinate transformation $Z = \sin\theta e^{i\phi}$.

The curvature singularity occurs at $\theta = \pi/2$. The procedure of fixing the gauge $\chi = 0$ and using the equations of motion to integrate the gauge field out is not valid near $\theta = \pi/2$, since it results into singular field configurations on the worldsheet. However, the full action (3.9) is perfectly well behaved at $\theta = \pi/2$. To see this, we expand the Lagrangian in (3.9) around $\theta = \pi/2$. Setting $\theta = \pi/2 - \tilde{\theta}$, we obtain that

$$S = \frac{-|k|}{2\pi}\int d^2z\phi F_{z\bar{z}} + O(\tilde{\theta}^2), \tag{3.11}$$

where we expressed the action in terms of manifestly gauge invariant degrees of freedom. The leading term in this expansion describes a simple topological theory, which shows that an alternative, non-geometric description of the theory can be given including the region near $\theta = \pi/2$. We return to this point later on.

From the form of the action near $\theta = \pi/2$, we also learn that the U(1) symmetry corresponding to shifts of the angle ϕ is quantum mechanically broken to a discrete symmetry $Z_{|k|}$. This is because compact worldsheets can support gauge field configurations for which $\int F_{z\bar{z}} = 2\pi in$, with n an integer, and such configurations must be summed over in the full path integral. It is clear then that the path integral is only invariant under discrete shifts of the angle ϕ: $\delta\phi = 2\pi m/|k|$.

This breaking of the classical U(1) symmetry to $Z_{|k|}$ is in accordance with the algebraic description of the SU(2)/ U(1) coset in terms of a system of $Z_{|k|}$ parafermionic currents

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$\psi_{\pm l}(z)$, $l = 0, 1 \ldots |k| - 1$ [with $\psi_0 = 1$, $\psi_l^\dagger \equiv \psi_{-l} = \psi_{k-l}$], of conformal weights $h_l = l(|k| - l)/|k|$. These satisfy the OPE relations

$$\begin{aligned} \psi_l(z)\psi_{l'}(0) &= c_{ll'} z^{-2ll'/|k|}(\psi_{l+l'}(0) + \ldots) \\ \psi_l(z)\psi_l^\dagger(0) &= z^{-2h_l}(1 + 2h_l z^2 T(0)/c + \ldots) \end{aligned} \tag{3.12}$$

which are invariant under the $Z_{|k|}$ global symmetry: $\psi_l \to e^{2\pi i l/k}\psi_l$. Here T is the energy momentum tensor of the parafermions, c the central charge (which is the same as the central charge of the coset model) and the coefficients $c_{ll'}$ are the parafermionic fusion constants [12]. In the infinite level $|k|$ limit, the conformal weights of the parafermion fields become integers. In this limit the sigma model metric is flat, and we recover the full rotational invariance of flat space [31]. The system can be also generalized to an $N = 2$ superconformal system by tensoring the $Z_{|k|}$ parafermions with a free compact boson as described in [34].

Finally we can check that the central charge remains the same after the analytic continuation. Indeed, it is the very fact that the central charge of the conformal field theory is smaller than the central charge corresponding to two macroscopic flat dimensions that codes the de Sitter nature of the two-dimensional cosmology.

4. The cosmological T-fold

The parafermionic T-fold

It is interesting to take a closer look at the geometry that we associate to the parafermionic model $\mathrm{SU}(2)_{|k|}/\,\mathrm{U}(1)$. As we already discussed, we describe it in terms of a metric and dilaton profile:

$$\begin{aligned} ds^2 &= |k|\alpha'(d\theta^2 + \tan^2\theta d\phi^2) \\ e^\Phi &= \frac{e^{\Phi_0}}{\cos\theta}, \end{aligned} \tag{4.1}$$

where $\phi \sim \phi + 2\pi$ and θ takes values in the interval $[0, \pi/2]$. This description breaks down near $\theta = \pi/2$. Nevertheless, the parafermionic conformal field theory is perfectly well-behaved, and we can wonder whether there is a more appropriate, almost-geometrical description. We argue that such a description exists in terms of a T-fold.

To obtain it, we perform a T-duality along the angular direction ϕ on the geometry described above:

$$\begin{aligned} ds^2 &= |k|\alpha' d\theta^2 + \frac{\alpha'}{|k|}\cot^2\theta d\tilde{\phi}^2 \\ e^\Phi &= \frac{e^{\tilde{\Phi}_0}}{\sin\theta}. \end{aligned} \tag{4.2}$$

By changing variables $\tilde{\theta} = \pi/2 - \theta$, we see that this is equivalent to:

$$\begin{aligned} ds^2 &= |k|\alpha' d\tilde{\theta}^2 + \frac{\alpha'}{|k|}\tan^2\tilde{\theta} d\tilde{\phi}^2 \\ e^\Phi &= \frac{e^{\tilde{\Phi}_0}}{\cos\tilde{\theta}}. \end{aligned} \tag{4.3}$$

This description is therefore at weak curvature (apart from an orbifold-like singularity) and weak coupling near $\theta = \pi/2$. Moreover, we can identify it as a $Z_{|k|}$ orbifold of a vectorially (or axially) gauged $\mathrm{SU}(2)/\mathrm{U}(1)$ coset. Indeed, it is true for the parafermionic theory that the T-dual and the $Z_{|k|}$ orbifold give two models with identical spectrum due to the coset character identity $\chi_{j,m} = \chi_{j,-m}$ (see e.g. [29, 30] for reviews).

We now use these facts to give an almost geometrical description of the parafermionic theory, in terms of a T-fold [19, 20]. We use the description in terms of the first geometry (4.1) near $\theta = 0$. We cut it just past $\theta = \pi/4$, where the radius of the circle is $\sqrt{|k|\alpha'}$. We glue it to the T-dual geometry which we consider near $\tilde{\theta} = 0$, or $\theta = \pi/2$, and which we cut just past $\tilde{\theta} = \pi/4$, where we have radius $\sqrt{\alpha'/|k|}$. We glue the circles (and their environments) using the T-duality transformation described above. In the gluing process, it is crucial to realize that we glue a patch with a direction of increasing radius to a T-dual patch which in the same direction has decreasing radius. That gives us the parafermionic T-fold. The associated partition function is (see e.g. [30] for a review):

$$Z = \sum_{j,m} \chi_{j,m}(\tau)\chi_{j,m}(\bar{\tau}). \tag{4.4}$$

One aspect of the model that is rendered manifest by the T-fold description is the breaking of the $\mathrm{U}(1)$ rotation symmetry to a discrete $Z_{|k|}$ symmetry, due to the $Z_{|k|}$ orbifolding. This is consistent with our previous discussion of the breaking due to worldsheet instantons. The T-fold yields an almost-geometrical picture of the symmetry breaking. The T-fold description is indeed everywhere regular modulo a benign orbifold singularity.

The cosmological T-fold

In the case of the two-dimensional cosmology as well, we can obtain a regular T-fold description of the target space of the conformal field theory. We recall that under T-duality (the metric can be obtained by analytically continuing the metric (4.2) in the direction $\tilde{\phi}$), the light-cone and the singularities get interchanged. Consider the cosmology, and cut it at a hyperbola at radius $\sqrt{|k|\alpha'}$, in between the light-cone and the time-like singularities in the Penrose diagram (see the upper part of figure 1). Consider then its T-dual, and cut it along a similar line. Glue the two parts of the T-dual cosmologies along these cuts to obtain the T-fold cosmology. The description we obtain is particularly nice as we no longer need a microscopic origin of a would-be source associated to the time-like singularities, nor do we need to define boundary conditions associated to them. There is no singularity in, nor is there a boundary to the T-fold cosmology. Indeed, the almost-geometrical description is very much like dS_2, which we can think of as a hyperboloid embedded in three-dimensional space. The difference is that the T-fold cosmology has two patches glued together via a T-duality transformation (instead of an ordinary coordinate transformation in the case of two-dimensional de Sitter space).

In figure 1 we show how the T-fold description of the parafermions and the two-dimensional cosmology continue into one another after analytic continuation.

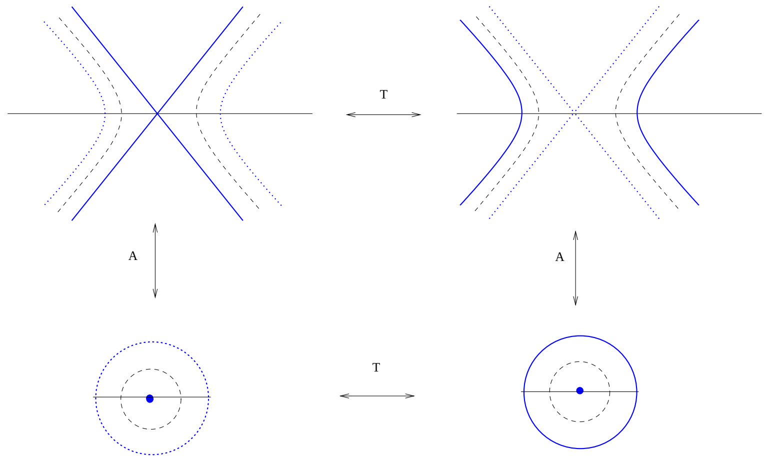

Figure 1: The analytic continuation of the T-fold. The upper part of the diagram shows the two T-dual descriptions of the cosmology in which the horizon and the (apparent) singularities are exchanged (in bold blue). The (striped black) cut along which they are glued is indicated, as well as the (thin black) line along which the cosmology is cut to obtain a space-like slice (see later). Analytic continuation then gives rise to the lower part of the figure, in which we have sketched the T-fold description of the parafermionic conformal field theory. In bold blue we have the center and the boundary of the disk, and (in black stripes) the T-dual circles along which we glue.

5. Defining the wave-function of the universe

Later on, we will consider string theory backgrounds which are product models and in which one factor consists of the two-dimensional cosmology discussed in sections 2, 3 and 4. For these models, we wish to define a wave-function of the universe in string theory following ideas of [1] which define a wave-function of dS_n universes within a field theoretic context.

We consider a time-reversal symmetric space-like slice of the cosmology, within the boundaries of the (seeming) singularities. See figure 2. This is the slice $T = 0$. In the past of the space-like slice, we glue half of the target space of an $\mathrm{SU}(2)/\,\mathrm{U}(1)$ coset conformal field theory — a half disk. By the analytic continuation discussed in the previous section, this gluing is continuous in the backgrounds fields, and moreover in the exact conformal field theory description.

A crucial feature of the proposal of [1] for the definition of the wave-function of the universe is that the corresponding Euclidean space is without boundary. In our set-up as well, the Euclidean conformal field theory has a target with no boundary. It is important in this respect that we have obtained an almost-geometric description of the parafermionic conformal field theory.[1] It is intuitively clear from the T-fold description given in the previous section that the parafermionic theory does not have a boundary.[2] When we cut the Euclidean T-fold into half, it is clear (from figure 1) that we can glue the boundary of

[1]A traditional description of the target space as a disk, which is singular,would lead to the faulty conclusion that the target space has a boundary.

[2]Since a T-fold is non-geometric, one needs to define the concept of boundary precisely. We believe that a reasonable definition will match our intuition.

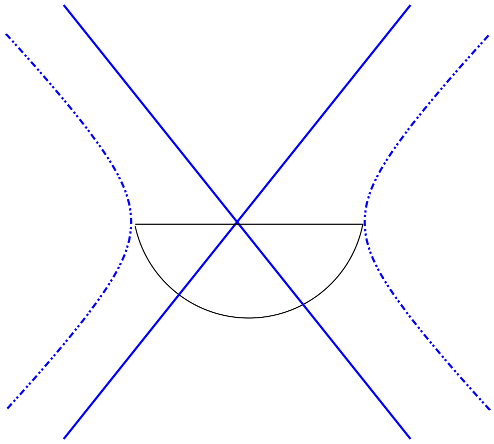

Figure 2: The continuous gluing of the half-disk into the cosmology, when cutting the cosmology along a space-like slice, and analytically continuing. The figure should be viewed as a simplified version of the previous T-fold picture.

that half-T-fold into the initial surface of the cosmological T-fold.

Thus we have determined the precise gluing of T-folds necessary in order to define a wave-function depending on initial data. We now define the wave-function of the universe by performing a "half T-fold" Euclidean path integral over *all* target space fields with specified values on the boundary:

$$\Psi[h_\partial, \phi_\partial, \dots] = \int [dg][d\phi] \dots e^{-S(g,\phi,\dots)} , \tag{5.1}$$

where the path integral is such that the metric, the dilaton and all other space-time fields satisfy $g = h_\partial, \phi = \phi_\partial, \dots$ on the boundary of the half T-fold that we glued into the cosmological solution. The path integral above can in principle be performed off-shell, in a second-quantized string field theory context, where we may also express it as an integral over a single string field Φ. (See e.g. [32] for a concise review).

Let us be more specific. The initial space-like slice of the T-fold cosmology has two patches. On each patch, we define a boundary metric h^1 and h^2, and similarly for other fields. The boundary metrics satisfy the condition that on the overlap of the patches, they match up to a T-duality transformation, symbolically: $h^1_{\partial^1 \cap \partial^2} = T(h^2_{\partial^1 \cap \partial^2})$. This is the way in which we can specify boundary data precisely. In the following, we do not emphasize this important part of the definition of the path integral further, not to clutter the formulas.

In principle, a T-fold path integral can be computed as follows. Consider again the two patches. Each patch has a non-singular geometric description. Over each patch the path integral reduces to an ordinary field theory path integral, and can be performed in the usual way giving rise to a functional of boundary data. The full path integral can be obtained by integrating the two functionals together over data that belong to the common boundary of the two patches. Since at the common boundary of the patches their fields are related by a T-duality transformation, to do the final integral we would need to perform a T-duality transformation on one of the two functionals.

We remark here that this particular feature of definitions of path integrals over T-folds with boundaries is generic. The above description is easily extended to a generic description of T-fold boundary data. Although we do not need a general prescription in this paper, we believe it would be interesting to develop the path integral formalism for T-folds with boundary further.

The prescription for the wave-function of the universe we outlined above should have an analogue, via the relation between string oscillators and the target space fields, to a first quantized prescription. Notice that the initial-time data allow multi closed string configurations. Summing over histories that lead to them, would allow worldsheets with boundaries (and other topology features), including disconnected ones. The wave-function would take the form

$$\Psi[X_\partial(\sigma,\tau)] = \sum_{\text{topologies}} \int [dX] e^{-S[X(\sigma,\tau)]}, \tag{5.2}$$

where the worldsheet path integrals are performed over string configurations $X(\sigma,\tau)$ that satisfy a specified boundary condition at given values of the zero-modes of the string configuration, i.e. at a given position of the target space. The equivalence of these descriptions is far from obvious, but it is made plausible by the fact that for two-dimensional string worldsheets, the first quantized description automatically comes with a prescription for the proper weighting of interaction vertices. The initial-time closed string configurations could be specified in terms of macroscopic loop operators discussed for example in [33]. The first quantized prescription considers fluctuations around a given background. A full second quantized prescription also integrates over backgrounds as in general relativity [1].

The wave-function so defined is hard to compute, although it may be obtained presumably for very particular boundary conditions. An example would be boundary conditions that are fixed by taking a Z_2 orbifold that folds over the disk onto itself — in that case, one may be able to compute the value of the wave-function for a particular argument. In order to understand better some global properties, we again follow [1] and concentrate on calculating the norm of the wave-function.

6. The norm of the wave-function

The norm of the wave-function is easier to compute. It is given by the following calculation:

$$\begin{aligned} ||\Psi||^2 &= \int [d\Phi_\partial] \int_{\text{half T-fold}} [d\Phi] e^{-S(\Phi)} \times \int_{\text{conj half T-fold}} [d\Phi] e^{-S(\Phi)} \\ &= \int_{\text{T-fold}} [d\Phi] e^{-S(\Phi)}, \end{aligned} \tag{6.1}$$

where we have expressed it as a string field theory path integral in terms of a string field Φ. The final integral is an integral over all possible string field configurations on the Euclidean T-fold. No boundary conditions need to be specified.

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We can do this calculation by considering the fluctuations around an on-shell closed string background, in a first quantized formalism:

$$||\Psi||^2 = \sum_{\text{topologies}} \int [dX] e^{-S[X(\sigma,\tau)]} \tag{6.2}$$

where $X(\sigma, \tau)$ is any mapping from the string worldsheet into the target space. The sum is over all closed worldsheet topologies, and includes a sum over disconnected diagrams. In fact it is equal to the following exponential of a sum of connected diagrams:

$$||\Psi||^2 = \exp(Z_{\text{total}}), \tag{6.3}$$

where the function Z_{total} is the total string theory partition function, which is defined as a sum over Euclidean worldsheet topologies:

$$Z_{\text{total}} = \frac{1}{g_s^2} Z_{S^2} + Z_{T^2} + g_s^2 Z_{genus=2} + \sum_{g=3}^{\infty} g_s^{2g-2} Z_{genus=g}. \tag{6.4}$$

Therefore, to evaluate the norm of the wave-function perturbatively, we need to evaluate the partition function for string theory on the Riemann surfaces of genus $0, 1, 2, \ldots$ and add their contributions with the appropriate power of the string coupling constant. The first contribution is akin to the tree level contribution in ordinary gravity, the second to the one-loop contribution, etc.

7. Thermal nature of the wave-function

A natural way to perform the Euclidean path integral in equation (5.1) over half the space is as follows. The origin $X = 0$ in one T-fold patch (and similarly for the other), divides the $T = 0$ slice into two halves: the left half corresponding to $X < 0$ and the right part corresponding to $X > 0$. We denote the boundary data on $X < 0$ by ϕ_R and on $X > 0$ by ϕ_L. See figure 3. By dividing the space into angular wedges spanning an overall angle equal to π, we can evaluate the path integral in terms of the generator of angular rotations. This generator is given by the analytic continuation of iH_ω, where $H_\omega = i\partial_\omega$ is the Hamiltonian conjugate to "Rindler" time in the region $uv > 0$ of the Lorentzian cosmology. Indeed in this region, we may set $u = \rho e^{-\omega}$, $v = \rho e^{\omega}$, with the string frame metric and dilaton given by

$$\begin{aligned} ds^2 &= |k|\alpha' \frac{d\rho^2 - \rho^2 d\omega^2}{1-\rho^2} \\ e^{2\Phi} &= \frac{e^{2\Phi_0}}{1-\rho^2}. \end{aligned} \tag{7.1}$$

In this patch, the background metric is static, invariant under time translations, and the dilaton field is space-like. Rotating to Euclidean signature amounts to setting $\omega = -i\phi$. So Rindler time translations correspond to angular rotations in the Euclidean space. As we

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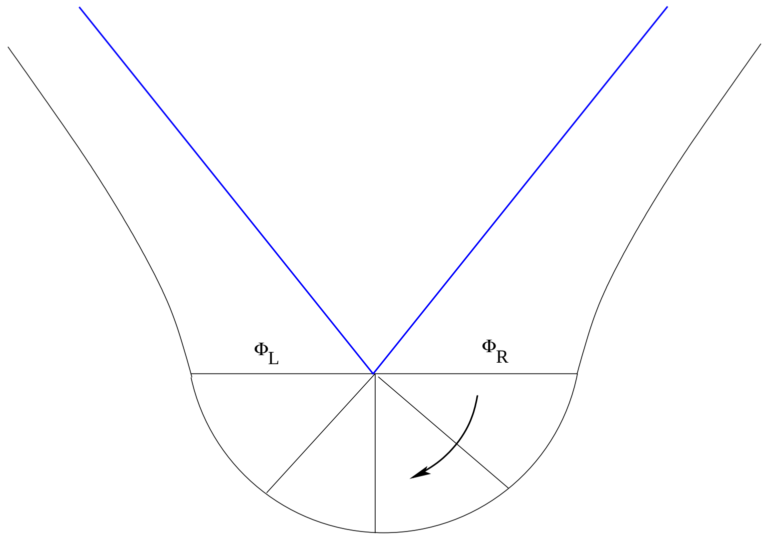

Figure 3: The thermal interpretation of the wave-function is obtained by thinking of the path integral as being performed along angular wedges from an initial (right) to a final (left) configuration.

have already discussed, only discrete angular rotations are true symmetries of the string theory background.

The boundary data can then be viewed as specifying initial *and* final conditions for the path integral evolution. This is clearly reviewed for the case of flat Rindler space and black hole spaces in [35]. In particular, the path integral measures the overlap between the data on the right ϕ_R, evolved for a Euclidean time π, and the data specified on the left ϕ_L (see figure 3), and it can be written as an amplitude

$$\Psi(\phi_L, \phi_R) = \langle \phi_L | e^{-\pi H_\omega} | \phi_R \rangle. \tag{7.2}$$

If we integrate over ϕ_L we obtain a thermal density matrix appropriate for the Rindler observer [35]

$$\int [d\phi_L] \Psi(\phi_L, \phi_R) \Psi^*(\phi_L, \phi_R') = \langle \phi_R' | e^{-2\pi H_\omega} | \phi_R \rangle, \tag{7.3}$$

with dimensionless temperature $T_\omega = 1/2\pi$. The norm of the wave-function is given by the trace

$$||\Psi||^2 = \mathrm{Tr} e^{-2\pi H_\omega} \tag{7.4}$$

and so it can be interpreted as a thermal space-time partition function. The genus-1 string contribution is a thermal one-loop amplitude.

In the full Euclidean path integral equation (6.1), the contributions of the fermionic fields are positive. To perform the full path integral over the whole T-fold, we divide it into angular wedges spanning an angle equal to 2π. Since the space has no non-contractible cycles, the space-time fermionic fields have to be taken anti-periodic in the angular variable and they contribute positively to the path integral.

The T-fold patches are glued along the hyperbola $\rho = 1/\sqrt{2}$ (see section 4). Near this region, the curvature is low for large enough level $|k|$. Thus for large level $|k|$, we may use the metric equation (7.1) to conclude that observers moving near the region $\rho \sim 1/\sqrt{2}$ measure a proper temperature $T \sim 1/(2\pi\sqrt{|k|\alpha'})$. In the cosmological region $uv \leq 0$, there is also an effective temperature of the same order as a result of particle production [14, 27]. For small level $|k|$, we need a string calculation to deduce the proper temperature of the system.

8. Specific examples

As we discussed above in order to derive the wave-function of the cosmology, we need to compute the total string partition function for the corresponding Euclidean background. When conformal field theories are compact, the genus-zero contribution to the total string partition function vanishes. This is because the string partition function is divided by the infinite volume of the conformal Killing group. This fact is a first important difference with the calculation in general relativity where the classical contribution is non-zero. In perturbative string theory the leading contribution is the genus-1 amplitude.

The Euclidean examples we shall describe here in detail belong to the family of $\hat{c} = 10$ superconformal, compact systems. In order for them to admit a Lorentzian continuation to a cosmological (time-dependent) background, space-time supersymmetry must be broken. Moreover, the models must be free of tachyons. The presence of tachyonic modes would indicate that the system undergoes a phase transition. The only known examples with the above properties are of the form

$$\frac{\mathrm{SU}(2)_{|k|}}{\mathrm{U}(1)} \quad \times \quad K \longleftrightarrow \quad \frac{\mathrm{SL}(2,R)_{-|k|}}{\mathrm{U}(1)} \quad \times \quad K\,, \tag{8.1}$$

where we indicated the analytic continuation from the Euclidean to the Lorentzian space-time. The level $|k|$ can be taken to be small.

As we already discussed, the relevant genus-1 string amplitude has to be thermal. The total superstring model has transverse central charge equal to $c = 12$ (or $\hat{c} = 8$). As a consequence, it has a Hagedorn transition at the fermionic radius $R_H = \sqrt{2\alpha'}$. In order for the genus-1 string amplitude to be finite, the physical temperature of the model has to be below the Hagedorn temperature: $T < T_H = 1/(2\pi\sqrt{2\alpha'})$. Let us give an argument that this can be realized for any $|k| \geq 2$. In writing the norm of the wave-function as a thermal space-time partition function, the role of the Hamiltonian is taken up by the generator of rotations on the disk. For the superconformal $\mathrm{SU}(2)/\,\mathrm{U}(1)$ model, the corresponding $\mathrm{U}(1)$ current is at level $|k|+2$ [34]. Thus we expect the physical temperature of the model to be set by the radius associated to this isometry generator, namely $\sqrt{(|k|+2)\alpha'}$. The corresponding temperature is given by

$$T = \frac{1}{2\pi\sqrt{(|k|+2)\alpha'}} \tag{8.2}$$

and is below the Hagedorn temperature for any positive (integer) level $|k|$. We will find that this temperature arises naturally in a level $|k| = 2$ model below. At level $|k| = 0$, where the minimal model has zero central charge, and consists only of the identity operator (and state), the cosmology disappears. When we reach the Hagedorn transition, the cosmology becomes so highly curved that it is no longer present in the string theory background.

8.1 Compact models

In a first class of specific examples that we will discuss in this section, we choose the level $|k| = 2$, and we take the internal conformal field theory K to be:

$$K = T^2 \times \prod_{i=1,\dots,7} \frac{\mathrm{SU}(2)_{k_i}}{\mathrm{U}(1)} \tag{8.3}$$

where all k_i's are taken equal to 2, so that $\hat{c}_K = 9$ (representing the central charge equivalent of nine flat directions). In the sequel, we set $\alpha' = 1$.

For this choice, the supersymmetric characters of the whole system are defined in terms of eight level $k = 2$ parafermionic systems (which are nothing but eight real fermions ψ_i and eight bosons ϕ_i compactified at the self-dual radius $R = 1$), and also a complex fermion Ψ_T and a complex boson Φ_T for the torus T^2. The $N = 2$ superconformal operators T_F, J are:

$$\begin{aligned} T_F &= \sum_{i=0,1,\dots 7} \psi_i \; e^{i\sqrt{2}\phi_i} \; + i\Psi_T \partial \Phi_T \\ J &= \frac{i}{\sqrt{2}} \sum_{i=0,1,\dots 7} \partial\phi_i + \Psi_T \bar{\Psi}_T . \end{aligned} \tag{8.4}$$

It is convenient to pair the $(0,1)$, $(2,3)$, $(4,5)$, $(6,7)$ systems respectively in order to obtain four copies of $\hat{c} = 2$ systems. For the first copy we define the bosons $\mathcal{H}_0$, $\mathcal{H}_1$ at radius $R = \sqrt{2}$ (or $R = \frac{1}{\sqrt{2}}$, fermionic T-dual points):

$$\phi_0 = \frac{1}{\sqrt{2}} \, (\mathcal{H}_0 + \mathcal{H}_1), \quad \phi_1 = \frac{1}{\sqrt{2}} \, (\mathcal{H}_0 - \mathcal{H}_1) \tag{8.5}$$

and similarly for the others. Then the currents are given by

$$\begin{aligned} T_F &= \sum_{i=0,2,4,6} \psi_i e^{i(\mathcal{H}_i + \mathcal{H}_{i+1})} + \psi_{i+1} \; e^{i(\mathcal{H}_i - \mathcal{H}_{i+1})} + i\partial\Phi_T e^{i\mathcal{H}_T} \\ J &= i\partial\mathcal{H}_0 + i\partial\mathcal{H}_2 + i\partial\mathcal{H}_4 + i\partial\mathcal{H}_6 + i\partial\mathcal{H}_T , \end{aligned} \tag{8.6}$$

where $i\partial\mathcal{H}_T = \Psi_T \bar{\Psi}_T$, which is also defined at the fermionic point. Observe that the $N = 2$ current is given in terms of $\mathcal{H}_0$, $\mathcal{H}_2$, $\mathcal{H}_4$, $\mathcal{H}_6$ and $\mathcal{H}_T$ *only* and is normalized correctly for a system with $\hat{c} = 10$.

The $N = 2$ left-moving characters of a particular $\hat{c} = 2$ system (e.g of the one containing $\mathcal{H}_0$), are expressed in terms of the usual level-2 Θ-functions:[3]

$$\frac{1}{\eta(\tau)^3} \; \Theta_{\psi_0} \begin{bmatrix} \gamma_0 \\ \delta_0 \end{bmatrix}^{\frac{1}{2}} \Theta_{\mathcal{H}_0} \begin{bmatrix} \alpha + 2H_0 \\ \beta + 2G_0 \end{bmatrix} \Theta_{\mathcal{H}_1} \begin{bmatrix} \gamma_0 - 2H_0 \\ \delta_0 - 2G_0 \end{bmatrix} \Theta_{\psi_1} \begin{bmatrix} 3\gamma_0 - 4H_0 \\ 3\delta_0 - 4G_0 \end{bmatrix}^{\frac{1}{2}} \tag{8.7}$$

[3]Our convention for the level-2 Θ function is $\Theta\left[{}^{\gamma}_{\delta}\right] = \sum_n \; e^{i\pi\tau(n+\frac{\gamma}{2})^2 + 2i\pi(n+\frac{\gamma}{2})(\frac{\delta}{2})}$.

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where the arguments (γ_0, δ_0) and $(4H_0, 4G_0)$ are integers. The later exemplifies the chiral Z_{k+2}-symmetry of the superconformal parafermionic characters ($k+2=4$ in our case). Similar expressions are obtained for the other three $\hat{c}=2$ parafermionic systems by replacing (γ_0, δ_0) with (γ_i, δ_i), and (H_0, G_0) by (H_i, G_i), $i=1,2,3$.

The global existence of the $N=2$ superconformal world-sheet symmetry and thus the existence of the left space-time supersymmetry imply

$$(H_T, G_T) + \sum_{i=0,1,2,3} (H_i, G_i) = \epsilon \bmod 2\,, \tag{8.8}$$

and similarly for the right supersymmetry. The arguments $(\epsilon, \bar{\epsilon})$ define the chirality of the space-time spinors. A simple choice is to set $(H_T, G_T) = (0,0)$, $(H_i, G_i) = (H, G)$ for $i=0,1$ and $(H_i, G_i) = (-H, -G)$ for $i=2,3$ (and similarly for the right arguments). Then if $(\epsilon, \bar{\epsilon}) = (1,1)$, space-time supersymmetry is broken. For the other choices there is some amount of supersymmetry preserved. In this class of models the only remaining possibility consistent with the global $N=2$ super-parameterization consists of shifts on Φ_T of the T^2 torus.

Z_4 orbifold models. Using the chiral Z_4 symmetry defined above and its subgroups, we can obtain four classes of models:

$$(H,G) = \frac{M}{4}(h,g), \quad \text{where } (h,g) = \text{ integers}, \quad M=1,2,3,4. \tag{8.9}$$

In particular, if we orbifold by Z_4 ($M=1$ or $M=3$), the ψ_0 parafermion decouples from the rest, especially from ψ_1. This is clear since in this case, the arguments of Θ_{ψ_0} and Θ_{ψ_1} become independent.

Initially, the arguments (γ_i, δ_i) are taken to be identical for the left- and right-moving characters. In this case the modular invariance of the partition function is manifest. Indeed, using the periodicity property of Θ-functions

$$\left|\Theta\begin{bmatrix}\gamma-2\epsilon\\ \delta-2\zeta\end{bmatrix}\right| = \left|\Theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\right|, \quad \epsilon,\zeta \text{ integers}, \tag{8.10}$$

and orbifolding by Z_4, ($M=1$), the genus-1 modular invariant partition function becomes:

$$\begin{aligned} Z = &\int_F \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2}\, \frac{\mathrm{Im}\tau}{\eta(\tau)^{12}\bar{\eta}(\bar{\tau})^{12}}\, \frac{1}{4}\sum_{h,g=0,1,2,3} \Gamma_{2,2} \times \frac{1}{2}\sum_{\gamma_0,\delta_0} \left|\Theta\begin{bmatrix}\gamma_0\\ \delta_0\end{bmatrix}\right| \left|\Theta\begin{bmatrix}\gamma_0-h\\ \delta_0-g\end{bmatrix}\right| \left|\Theta\begin{bmatrix}\gamma_0-\frac{h}{2}\\ \delta_0-\frac{g}{2}\end{bmatrix}\right|^2 \\ &\times \frac{1}{2}\sum_{\gamma_1,\delta_1} \left|\Theta\begin{bmatrix}\gamma_1\\ \delta_1\end{bmatrix}\right| \left|\Theta\begin{bmatrix}\gamma_1-h\\ \delta_1-g\end{bmatrix}\right| \left|\Theta\begin{bmatrix}\gamma_1-\frac{h}{2}\\ \delta_1-\frac{g}{2}\end{bmatrix}\right|^2 \times \frac{1}{2}\sum_{\gamma_2,\delta_2} \left|\Theta\begin{bmatrix}\gamma_2\\ \delta_2\end{bmatrix}\right| \left|\Theta\begin{bmatrix}\gamma_2+h\\ \delta_2+g\end{bmatrix}\right| \left|\Theta\begin{bmatrix}\gamma_2+\frac{h}{2}\\ \delta_2+\frac{g}{2}\end{bmatrix}\right|^2 \\ &\times \frac{1}{2}\sum_{\gamma_3,\delta_3} \left|\Theta\begin{bmatrix}\gamma_3\\ \delta_3\end{bmatrix}\right| \left|\Theta\begin{bmatrix}\gamma_3+h\\ \delta_3+g\end{bmatrix}\right| \left|\Theta\begin{bmatrix}\gamma_3+\frac{h}{2}\\ \delta_3+\frac{g}{2}\end{bmatrix}\right|^2 \times \frac{1}{2}\sum_{(\alpha,\beta)} e^{i\pi(\alpha+\beta+\epsilon\alpha\beta)}\ \Theta\begin{bmatrix}\alpha+\frac{h}{2}\\ \beta+\frac{g}{2}\end{bmatrix}^2 \Theta\begin{bmatrix}\alpha-\frac{h}{2}\\ \beta-\frac{g}{2}\end{bmatrix}^2 \\ &\times \frac{1}{2}\sum_{(\bar{\alpha},\bar{\beta})} e^{i\pi(\bar{\alpha}+\bar{\beta}+\bar{\epsilon}\bar{\alpha}\bar{\beta})}\ \bar{\Theta}\begin{bmatrix}\bar{\alpha}+\frac{h}{2}\\ \bar{\beta}+\frac{g}{2}\end{bmatrix}^2 \bar{\Theta}\begin{bmatrix}\bar{\alpha}-\frac{h}{2}\\ \bar{\beta}-\frac{g}{2}\end{bmatrix}^2 . \end{aligned} \tag{8.11}$$

The arguments (α, β) and $(\bar{\alpha}, \bar{\beta})$ are those associated to the $N=2$ left- and right-moving supercurrents. If $(\epsilon, \bar{\epsilon}) = (1,1)$, supersymmetry is broken. For all other choices this partition function is identically zero and there is some amount of supersymmetry preserved. To obtain the above result we have used the fact that the contribution of the superconformal ghosts cancel the oscillator contributions of the T^2 supercoordinates (Φ_T, Ψ_T). This is the reason for choosing the Z_4 not to act on Ψ_T, having set $(H_T, G_T) = (0,0)$. The only remnant from the torus contribution is the $\Gamma_{2,2}$ lattice, which can be possibly shifted by $(Lh/2, Lg/2)$ with either $L = 0$ or $L = 1, 2, 3$ (as we will see later).

To proceed we need to identify and insert the thermal co-cycle $S\begin{bmatrix} q, & (\alpha+\bar{\alpha}) \\ p, & (\beta+\bar{\beta}) \end{bmatrix}$ associated to the time direction of the cosmology:

$$S\begin{bmatrix} q, & (\alpha+\bar{\alpha}) \\ p, & (\beta+\bar{\beta}) \end{bmatrix} = e^{i\pi\left(p(\alpha+\bar{\alpha}) + q(\beta+\bar{\beta}) \right)}, \tag{8.12}$$

where p and q are the lattice charges associated to the compactified Euclidean time. Here, $F_\alpha = (\alpha + \bar{\alpha})$ and $F_\beta = (\beta + \bar{\beta})$ define the spin of space-time particles: $F_\alpha = 1$ modulo 2 for fermions and $F_\alpha = 0$ modulo 2 for bosons. To impose this co-cycle insertion, it is necessary to rewrite the partition function in a form that reveals the charge lattice (p, q) of the Euclidean time direction. To this end, it is convenient to separate the partition function into the "untwisted sector" $(h, g) = (0, 0)$ and " twisted sectors" $(h, g) \neq (0, 0)$,

$$Z = \int_F \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2} \left(Z_{unt} + \sum_{(h,g)\neq(0,0)} Z_{\text{twist}} \begin{bmatrix} h \\ g \end{bmatrix} \right). \tag{8.13}$$

To isolate the relevant (p, q) charge lattice, we use the identity

$$\left|\Theta \begin{bmatrix} \gamma \\ \delta \end{bmatrix}\right|^2 = \frac{R}{\sqrt{\mathrm{Im}\tau}} \sum_{(m,n)} e^{-\pi R^2 \frac{|m+\tau n|^2}{\mathrm{Im}\tau}} \, e^{i\pi(m\gamma+n\delta+mn)}, \qquad R^2 = \frac{1}{2}. \tag{8.14}$$

Although in the above identity the radius is fixed to the fermionic point $R^2 = 1/2$, we note that *the modular transformation properties are the same for any* R^2, and in particular for the dual-fermionic point with $R^2 = 2$.

Using the above identity for the conformal block involving the parafermions $\psi_{0,1}$ and the $\mathcal{H}_1$ field, we obtain

$$\frac{1}{2}\sum_{(\gamma,\delta)} \left|\Theta_{\psi_{0,1}} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}\right|^2 \left|\Theta_{\mathcal{H}_1} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}\right|^2 = \frac{1}{2}\sum_{(\gamma,\delta)} \sum_{(m_1,n_1),(m_2,n_2)} \frac{R_1 R_2}{\mathrm{Im}\tau} e^{-\pi R_1^2 \frac{|m_1+\tau n_1|^2}{\mathrm{Im}\tau} - \pi R_2^2 \frac{|m_2+\tau n_2|^2}{\mathrm{Im}\tau}} \times e^{i\pi((m_1+n_1)\gamma + (m_2+n_2)\delta + m_1 n_1 + m_2 n_2)} \tag{8.15}$$

where $R_1^2 = R_2^2 = R^2 = 1/2$. Since the arguments (γ, δ) do not appear elsewhere (in equation (8.11)), summing over them forces $(m_1 + m_2)$ and $(n_1 + n_2)$ to be even integers. This constraint can be solved if we take

$$m_1 = p_1 + p_2, \quad m_2 = p_1 - p_2, \quad n_1 = q_1 + q_2, \quad n_2 = q_1 - q_2 \tag{8.16}$$

so that,

$$\frac{1}{2}\sum_{(\gamma,\delta)} \left|\Theta_{\psi_{0,1}} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}\right|^2 \left|\Theta_{\mathcal{H}_1} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}\right|^2 = \Gamma_{1,1}(R_+)\Gamma_{1,1}(R_-) =$$

$$e \left(\sum_{(p_1,q_1)} \mathrm{Im}\tau^{-\frac{1}{2}} R_+ e^{-\pi R_+^2 \frac{|p_1+\tau q_1|^2}{\mathrm{Im}\tau}} \right) \left(\sum_{(p_2,q_2)} \mathrm{Im}\tau^{-\frac{1}{2}} R_- e^{-\pi R_-^2 \frac{|p_2+\tau q_2|^2}{\mathrm{Im}\tau}} \right) \tag{8.17}$$

with $R_+^2 = R_-^2 = 2R^2 = 1$. Therefore, the partition function of the *bosonic* part of parafermions *factorizes in two* $\Gamma_{1,1}$ lattices, both of them with twice the initial radius squared.

The charge lattices (p_1, q_1) and (p_2, q_2) are associated to the ψ_0 and ψ_1 parafermions. We can see this as follows. Consider the left-charge operators which are well defined in the untwisted sector:

$$Q_+ = i \oint dz(\psi_0\psi_1 + \partial\mathcal{H}_1), \qquad Q_- = i \oint dz(\psi_0\psi_1 - \partial\mathcal{H}_1) \tag{8.18}$$

and similarly for the right-moving ones $(\bar{Q}_\pm)$. Then,

$$\begin{aligned}
(Q+\bar{Q})_+ &= m_1 + m_2 = 2p_1, \qquad (Q-\bar{Q})_+ = n_1 + n_2 = 2q_1 \\
(Q+\bar{Q})_- &= m_1 - m_2 = 2p_2, \qquad (Q-\bar{Q})_- = n_1 - n_2 = 2q_2
\end{aligned} \tag{8.19}$$

where we have used the constraint (8.16). We identify the charges (p_1, q_1) as the momenta that enter in the thermal co-cycle, and associate the lattice to the Euclidean time direction.

Before we proceed further, let us stress the following point. We started with a diagonal modular invariant combination and with initial radii $R_1^2 = R_2^2 = R^2 = 1/2$. The anti-diagonal choice implies that the initial values for the radii are at the fermionic T-dual points, namely $R_1^2 = R_2^2 = R^2 = 2$. Notice that in all we perform two T-dualities simultaneously so that we remain in the same type II theory. Thus the conformal block, equation (8.15), can be replaced with the T-dual one with $R^2 = 2$.[4] For the anti-diagonal choice, the radii of the corresponding factorized lattices are given by $R_+^2 = R_-^2 = 4$ instead of unity for the diagonal combination.

Thus in the untwisted sector, $Z_{\rm unt}$ has to be replaced with

$$Z_{\rm unt} \longrightarrow Z_{\rm unt}^{\rm thermal} = \sum_{(p_1,q_1)} \sum_{(\alpha,\beta),(\bar\alpha,\bar\beta)} Z_{\rm unt}^{\rm thermal}\begin{bmatrix} q_1, \ \alpha+\bar\alpha \\ p_1, \ \beta+\bar\beta \end{bmatrix} e^{i\pi\left(p_1(\alpha+\bar\alpha)+q_1(\beta+\bar\beta)\right)}. \tag{8.20}$$

Performing a similar factorization for the remaining three copies of $\hat{c} = 2$ superconformal parafermionic blocks, we can write the thermal untwisted contribution in a compact form:

$$Z_{\rm unt}^{\rm thermal} = \frac{1}{4}\frac{{\rm Im}\tau\Gamma_{9,9}}{\eta^{12}\bar\eta^{12}} \left\{ \frac{1}{4} \sum_{(\alpha,\beta),(\bar\alpha,\bar\beta)} \Gamma_{1,1}(R_+) \begin{bmatrix} \alpha+\bar\alpha \\ \beta+\bar\beta \end{bmatrix} e^{i\pi(\alpha+\beta+\epsilon\alpha\beta)}\Theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix}^4 e^{i\pi(\bar\alpha+\bar\beta+\bar\epsilon\bar\alpha\bar\beta)}\bar\Theta\begin{bmatrix} \bar\alpha \\ \bar\beta \end{bmatrix}^4 \right\}. \tag{8.21}$$

The $\Gamma_{9,9}$ lattice factor is composed of a product of lattices: the initial $\Gamma_{2,2}$ lattice of the torus T^2 with radii R_y, R_z, the $\Gamma_{1,1}$ lattice of the first parafermionic block at radius R_-, and the product of three pairs $\Gamma_{1,1}(R_+^i)\,\Gamma_{1,1}(R_-^i)$ for the other three parafermionic blocks. The $\Gamma_{1,1}(R_+)$ lattice is the thermally shifted lattice

$$\Gamma_{1,1}(R_+)\begin{bmatrix} \alpha+\bar\alpha \\ \beta+\bar\beta \end{bmatrix} = \sum_{(p_1,q_1)} {\rm Im}\tau^{-\frac{1}{2}} R_+ e^{-\pi R_+^2 \frac{|p_1+\tau q_1|^2}{{\rm Im}\tau}} e^{i\pi\left((\alpha+\bar\alpha)p_1+(\beta+\bar\beta)q_1\right)}. \tag{8.22}$$

[4] In Gepner's formalism [34] the anti-diagonal combination corresponds to exchanging $(m, \bar{m}) \to (m, -\bar{m})$.

Its coupling with the space-time spin structure breaks space-time supersymmetry so that both bosons and fermions give positive contributions to the thermal partition function. The radius R_+ sets the temperature of the system: $2\pi T = 1/R_+$. The form of the thermal coupling in equation (8.21) is similar to the one that appears in the familiar flat type II superstring theories at finite temperature. The difference here is that the temperature is *fixed*. Since we have succeeded to factorize out the thermal lattice, we can now treat all other radii parameterizing the $\Gamma_{9,9}$ lattice as independent moduli. To obtain the four dimensional interpretation we discussed in section 2, we take the radii $R_{y,z}$ to be large keeping all other ones small.

As we already remarked, there are only two choices consistent with the cosmological interpretation of the partition function corresponding to the two values of the radius R_+. For the diagonal choice we have a radius $R_+^2 = 1$ corresponding to a temperature $2\pi T = 1/R_+$ higher than Hagedorn: $2\pi T_H = 1/R_H = 1/\sqrt{2}$. This model is tachyonic and so unstable in perturbation theory. For the second anti-diagonal choice $R_+^2 = 4$, and the temperature is *below* Hagedorn: $2\pi T = 1/2 < 2\pi T_H$. This is precisely the temperature that we gathered from general arguments, equation (8.2) for level $|k| = 2$. That model gives rise to a well defined, integrable partition function and a finite norm for the wave-function at one loop. The integral is difficult to perform analytically but it can be estimated. We will not carry out this computation here.

The remaining part consists of the twisted sectors of the theory, $(h, g) \neq 0$. Here we shall find new stringy phenomena associated with the fact that we are orbifolding the Euclidean time circle. Since this is twisted, the thermal co-cycle has to be extended consistently. It takes the general form, valid for both the untwisted and the twisted sectors:

$$S\begin{bmatrix} (q+h), & (\alpha+\bar{\alpha}) \\ (p+g), & (\beta+\bar{\beta}) \end{bmatrix} = e^{i\pi\left(\,(p+g)(\alpha+\bar{\alpha})\,+\,(q+h)(\beta+\bar{\beta})\,\right)}. \tag{8.23}$$

That is, the relevant lattice is augmented by the quantum numbers (g, h) that label the twisted sectors. Again, the thermal co-cycle insures that fermions contribute positively to the partition function. In the twisted sectors, there is no momentum charge and we can set $(p, q) = (0, 0)$.

In the twisted sectors, each of the $\hat{c} = 2$ superconformal blocks is equivalent to a system described by a free complex boson and a free complex fermion twisted by Z_4. This equivalence implies topological identities for each $N = 2$ twisted superconformal block [36, 37]:

$$\frac{1}{2|\eta|^4}\sum_{(\gamma_i,\delta_i)}\left|\Theta\begin{bmatrix}\gamma_i\\ \delta_i\end{bmatrix}\right|\ \left|\Theta\begin{bmatrix}\gamma_i-h\\ \delta_i-g\end{bmatrix}\right|\ \left|\Theta\begin{bmatrix}\gamma_i-\frac{h}{2}\\ \delta_i-\frac{g}{2}\end{bmatrix}\right|^2 = 2^2\,\sin^2\!\left(\frac{\pi\Lambda(h,g)}{4}\right)\,\frac{|\eta|^2}{\left|\Theta\begin{bmatrix}1+\frac{h}{2}\\ 1+\frac{g}{2}\end{bmatrix}\right|^2} \tag{8.24}$$

where $\Lambda(h, g) = \Lambda(g, h)$ depend on the (h, g)-twisted sector. $\Lambda(h, g) = 2$ when $(h, g) =$(0,2), (2,0) and (2,2) while for the remaining 12 twisted sectors $\Lambda(h, g) = 1$. Although the above orbifold expressions are derived at the fermionic point, they remain valid for any other point of the untwisted moduli space.

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Using the above orbifold identity, the "twisted" part of the thermal partition function simplifies to:

$$Z^{\text{thermal}}_{\text{twist}} = \frac{1}{4}\sum_{(\alpha,\beta,\bar\alpha,\bar\beta)} \frac{1}{4}\sum_{(h,g)\neq(0,0)} \text{Im}\tau\ \Gamma_{2,2}\ e^{i\pi(\alpha+\beta+\epsilon\alpha\beta)} e^{i\pi(\bar\alpha+\bar\beta+\bar\epsilon\bar\alpha\bar\beta)} e^{i\pi(\alpha+\bar\alpha)g+i\pi(\beta+\bar\beta)h}$$

$$\times\ 2^8\ \sin^8(\frac{\pi\Lambda(h,g)}{4})\ \frac{\Theta\begin{bmatrix}\alpha+\frac{h}{2}\\ \beta+\frac{g}{2}\end{bmatrix}^2 \Theta\begin{bmatrix}\alpha-\frac{h}{2}\\ \beta-\frac{g}{2}\end{bmatrix}^2\ \bar\Theta\begin{bmatrix}\bar\alpha+\frac{h}{2}\\ \bar\beta+\frac{g}{2}\end{bmatrix}^2\ \bar\Theta\begin{bmatrix}\bar\alpha-\frac{h}{2}\\ \bar\beta-\frac{g}{2}\end{bmatrix}^2}{\left|\Theta\begin{bmatrix}1+\frac{h}{2}\\ 1+\frac{g}{2}\end{bmatrix}\Theta\begin{bmatrix}1-\frac{h}{2}\\ 1-\frac{g}{2}\end{bmatrix}\right|^4}\,. \tag{8.25}$$

Furthermore, by using the left- (and right-) Jacobi identities

$$\frac{1}{2}\sum_{(\alpha,\beta)} e^{i\pi(\alpha+\beta+\epsilon\alpha\beta)}\Theta\begin{bmatrix}\alpha+\frac{h}{2}\\ \beta+\frac{g}{2}\end{bmatrix}^2 \Theta\begin{bmatrix}\alpha-\frac{h}{2}\\ \beta-\frac{g}{2}\end{bmatrix}^2 = -|\epsilon|\Theta\begin{bmatrix}1+\frac{h}{2}\\ 1+\frac{g}{2}\end{bmatrix}^2 \Theta\begin{bmatrix}1-\frac{h}{2}\\ 1-\frac{g}{2}\end{bmatrix}^2, \tag{8.26}$$

the twisted part of the thermal partition function simplifies further:

$$Z^{\text{thermal}}_{\text{twist}} = \frac{1}{4}\sum_{(h,g)\neq(0,0)} \text{Im}\tau\ \Gamma_{2,2}\ 2^8\ \sin^8(\frac{\pi\Lambda(h,g)}{4})\ \ \phi((g,h),(\epsilon,\bar\epsilon)). \tag{8.27}$$

The factor ϕ depends on the initial choice of the left- and right-chirality coefficients $\epsilon, \bar\epsilon$

$$\phi((g,h),(\epsilon,\bar\epsilon)) = \Big(|\epsilon| + (1-|\epsilon|)(1-e^{i\pi gh})\Big)\ \Big(|\bar\epsilon| + (1-|\bar\epsilon|)(1-e^{i\pi gh})\Big)\,. \tag{8.28}$$

It should be noted that expression (8.27) has a field theoretic interpretation in terms of a momentum lattice only.

If the initial non-thermal model was not supersymmetric, $|\epsilon| = |\bar\epsilon| = 1$, then the number of massless bosons and fermions will not be equal, $n_b \neq n_f$. For all other choices $n_b = n_f$. This situation is reflected in the factor $\phi((g,h),(\epsilon,\bar\epsilon))$, which distinguishes the four different possibilities. In the non supersymmetric case there is a non-vanishing contribution to the partition function even in the absence of the thermal co-cycle. This is equivalent, in field theory, to the one-loop *zero temperature contribution to the effective action.* This contribution is zero in supersymmetric theories. In the later cases, the corrections are coming from the massive thermal bosons and fermions plus a contribution from massless bosons. We will display here a number of typical examples. The first class is when the $\Gamma_{2,2}$ lattice is unshifted by (h,g) and so it factorizes out from the sum over (g,h).

Unshifted $\Gamma_{2,2}$ lattice. In the unshifted case, the sum over (g,h) can be performed easily so that the only remaining dependence is that of $\epsilon, \bar\epsilon$. We obtain [38, 39],

$$S^{\text{thermal}}_{\text{twist}} = \int_F \frac{d\tau d\bar\tau}{(\text{Im}\tau)^2} Z^{\text{thermal}}_{\text{twist}} = -C[\epsilon,\bar\epsilon]\ \log\left[\ |\eta(T)|^4\ |\eta(U)|^4\ \text{Im}T\ \text{Im}U\mu^2\ \right] \tag{8.29}$$

where μ^2 is an infrared cut-off and T, U parameterize the Kähler and complex structure moduli of the target space torus T^2. The coefficient $C[\epsilon,\bar\epsilon]$ depends on the initial chiralities of the spinors:

$$C[1,1] = 240, \qquad C[1,0] = C[0,1] = 32, \qquad C[0,0] = 64\,. \tag{8.30}$$

Actually, $C[\epsilon,\bar{\epsilon}]$ is nothing but the number of the massless bosonic degrees of freedom of the theory. Equation (8.29) is invariant under the full target space T-duality group acting on the T and U moduli separately.

For large volume, $\mathrm{Im}T \gg 1$, the leading behavior is linear in $\mathrm{Im}T \sim R_y R_z$. Assuming $iU \sim R_y/R_z$ fixed and $\mu^2 \sim \gamma/R_y R_z$, we have

$$S^{\rm thermal}_{\rm twist} \;=\; C[\epsilon,\bar{\epsilon}]\,\left(\frac{\pi}{3}\,R_y R_z - \log\left[\,\gamma\,|\eta(U)|^4\ \mathrm{Im}U\,\right]\right). \tag{8.31}$$

Note that in the large volume limit, the twisted sector contribution to one-loop amplitude depends both on the Kähler and complex structure moduli.

Shifted lattice. Another illustrating example is when the Z_4 action shifts the $\Gamma_{2,2}$ lattice simultaneously with the twist we described before. In this case the lattice is replaced by a shifted $\Gamma_{2,2}$ lattice. The $(Lh/2, Lg/2)$ shifted lattice, $L = 1,2,3$, is given by:[5]

$$\Gamma_{1,1}\begin{bmatrix}\frac{Lh}{2}\\ \frac{Lg}{2}\end{bmatrix}(R) = \sum_{(m,n)} \frac{R}{\sqrt{\mathrm{Im}\tau}}\ e^{-\pi R^2 \frac{|(4m+Lg)+(4n+Lh)\tau|^2}{\mathrm{Im}\tau}}. \tag{8.32}$$

Here, we will examine in more detail the $L = 1$ case which corresponds to a 1/4-shifted lattice.

When $(\epsilon,\bar{\epsilon}) = (1,1)$, the contribution of the twisted sector to the partition function becomes [37, 40]:

$$S^{\rm thermal}_{\rm twist} \;=\; 240\ \int_F \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2}\ \mathrm{Im}\tau\left(\ \Gamma_{2,2}[T,U] - \frac{1}{4}\Gamma_{2,2}[4T,4U]\ \right). \tag{8.33}$$

To obtain the above expression we have used the identities

$$\sum_{(h,g)} \Gamma_{2,2}\begin{bmatrix}h/2\\ g/2\end{bmatrix} = \Gamma_{2,2}(T,U), \qquad \Gamma_{2,2}[\,{}^0_0\,] = \frac{1}{4}\Gamma_{2,2}[4T,4U] \tag{8.34}$$

and we subtracted the contribution of the untwisted sector, $(g,h) = (0,0)$. Integrating over τ we obtain [37, 40]

$$S^{\rm thermal}_{\rm twist} \;=\; -\,60\ \log\left[\frac{|\eta(T)|^{16}\ |\eta(U)|^{16}}{|\eta(4T)|^4\ |\eta(4U)|^4}\ \frac{\mathrm{Im}T^3\ \mathrm{Im}U^3\mu^6}{16}\right]. \tag{8.35}$$

There is no volume factor in the large $\mathrm{Im}T$ limit (and this is generic in the case of freely acting orbifolds [37, 40]). So for large $\mathrm{Im}T$, (and setting $\mathrm{Im}T\mu^2 \sim \gamma$), we obtain

$$S^{\rm thermal}_{\rm twist} = -60\ \log\left[\gamma\frac{|\eta(U)|^{16}}{|\eta(4U)|^4}\ \frac{\mathrm{Im}U^3}{16}\right]. \tag{8.36}$$

In the large volume limit, $S^{\rm thermal}_{\rm twist}$ only depends on the complex structure modulus of the torus.

[5] For brevity we have given the expression for the shifted $\Gamma_{1,1}$ lattice but the generalization to the $\Gamma_{2,2}$ lattice is straightforward.

Comments. The total Euclidean one-loop amplitude in both the shifted and unshifted lattice cases is given by:

$$S^{\text{thermal}} = S^{\text{thermal}}_{\text{unt}} + S^{\text{thermal}}_{\text{twist}} \tag{8.37}$$

The $S^{\text{thermal}}_{\text{unt}}$ is nothing but one-quarter of the thermal partition function of type II superstring theory on $S^1 \times T^9$ with all nine spatial radii arbitrary, while that of Euclidean time fixed by the temperature: $2\pi T = 1/R_+ = 1/2$. By exponentiating S^{thermal}, with the twisted sector contributions in our examples given by equations (8.31) and (8.36), we obtain the norms of the corresponding cosmological wave-functions as functions of all moduli.

The difference between the shifted and the ordinary model discussed previously can be understood as follows. It is known that the freely acting orbifolds are related to gravitational and gauge field backgrounds with fluxes [41]. This indicates the different interpretations of the two cosmological models with shifted and unshifted $\Gamma_{2,2}$. In the shifted model there are non-vanishing magnetic fluxes [41] while in the unshifted case such fluxes are absent.

Let us stress here that the thermal Z_4-orbifold described involves a twisting that leaves two moduli parameterizing a $\Gamma_{2,2}$ lattice, which in the large moduli limit gives us the four dimensional cosmological model discussed in section 2. Many other orbifold-like models can be constructed, which may factorize bigger lattices, admitting a higher dimensional interpretation. In all those cases, the partition function is computable as a function of the moduli. However its analytic form in terms of the moduli in various limits depends crucially on whether the orbifold is freely, or even partially freely acting (or in other words, on the different structure of magnetic fluxes). Models based on asymmetric orbifolds can also be constructed giving rise to a rich family of calculable models. A more detailed analysis would be interesting so as to understand the classification of the low level cosmologies, as well as the characteristic dependence of the norm of the wave-function on the various features of the large class of models. To illustrate the above points, we offer one further simple example based on a Z_2 instead of the Z_4 orbifold.

Z_2 orbifold models. In the Z_2 orbifold models ($M = 2$ in equation (8.9)), the factorization of the cosmological CFT factor is not so explicit as it was in the Z_4 examples. However the cosmological interpretation remains the same.

For the untwisted sector, the genus-1 contribution $S^{\text{thermal}}_{\text{unt}}$ is now one half of the thermal partition function of type II theory on $S^1 \times T^9$. We proceed to analyze the twisted sector contribution to the genus-1 amplitude. Following similar steps as in the Z_4 orbifold case, and now setting $(2H, 2G) = (h, g)$ to be integers defined modulo 2, we obtain for the case $(\epsilon, \bar{\epsilon}) = (0, 0)$

$$S^{\text{thermal}}_{\text{twist}} = \int_F \frac{d\tau d\bar{\tau}}{(\text{Im}\tau)^2} \, \frac{1}{2} \sum_{(h,g)\neq(0,0)} 2^8 \, \text{Im}\tau \; \Gamma_{2,2} \; . \tag{8.38}$$

To arrive at the result notice that since the characters (h, g) are defined to be integers modulo 2, the factor of $1/4$ in the first line of equation (8.11) now becomes $1/2$; no other modifications in this formula are needed. All the other steps carry through as before. Here also we may classify the models in two classes; in the first class the lattice is taken to be unshifted, while in the second class the $\Gamma_{2,2}$ is half-shifted. We will use the definition:

$$\Gamma_{1,1}\left[{}^{Lh}_{Lg}\right](R) = \sum_{(m,n)} \frac{R}{\sqrt{\mathrm{Im}\tau}}\, e^{-\pi R^2 \frac{|(2m+Lg)+(2n+Lh)\tau|^2}{\mathrm{Im}\tau}}. \tag{8.39}$$

For the *unshifted* $\Gamma_{2,2}$ Z_2-model, and when $(\epsilon, \bar{\epsilon}) = (0,0)$, we obtain:

$$S^{\rm thermal}_{\rm twist} = -384\ \log\left[|\eta(T)|^4\ |\eta(U)|^4\ \mathrm{Im}T\ \mathrm{Im}U\mu^2\right]. \tag{8.40}$$

As in the Z_4 models, for large volume, $\mathrm{Im}T \gg 1$, the leading behavior is linear in $\mathrm{Im}T \sim R_y R_z$. Assuming $iU \sim R_y/R_z$ fixed and $\mu^2 \sim \gamma/R_y R_z$ we have:

$$S^{\rm thermal}_{\rm twist} = 384\ \left(\frac{\pi}{3}\ R_y R_z - \log\left[\ \gamma\ |\eta(U)|^4\ \mathrm{Im}U\ \right]\right). \tag{8.41}$$

The Z_2-model with *half-shifted* lattice, $\Gamma_{2,2}[^h_g]$, $(g,h=0,1)$, yields [37, 40]

$$S^{\rm thermal}_{\rm twist} = 384\ \int_F \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2}\ \mathrm{Im}\tau \left(\ \Gamma_{2,2}[T,U] - \frac{1}{2}\Gamma_{2,2}[\,2T,2U]\ \right). \tag{8.42}$$

Here also the contribution coming from the untwisted sector $(h,g) = (0,0)$, is subtracted. To obtain the above expression we have used the identities

$$\sum_{(h,g)} \Gamma_{2,2}[^{\,h}_{\,g}\,] = \Gamma_{2,2}[T,U], \qquad \Gamma_{2,2}[^{\,0}_{\,0}\,] = \frac{1}{2}\Gamma_{2,2}[\,2T,2U]. \tag{8.43}$$

Integrating over τ we obtain:

$$S^{\rm thermal}_{\rm twist} = -192\ \log\left[\frac{|\eta(T)|^8\ |\eta(U)|^8}{|\eta(2T)|^4\ |\eta(2U)|^4}\ \frac{\mathrm{Im}T\ \mathrm{Im}U\mu^2}{4}\right]. \tag{8.44}$$

There is no volume factor in the large $\mathrm{Im}T$ limit. Thus, for large $\mathrm{Im}T$, (and setting $\mathrm{Im}T\mu^2 \sim \gamma$), we obtain for the half-shifted lattice contribution to $S^{\rm thermal}_{\rm twist}$:

$$S^{\rm thermal}_{\rm twist} = -192\ \log\left[\frac{\gamma\ |\eta(U)|^8}{|\eta(2U)|^4}\ \ \frac{ImU}{4}\right]. \tag{8.45}$$

As we see, we can obtain explicit expressions for the genus-1 approximation to the norm of the wave-function for these particular Z_2-models. The twisted sector contribution is given explicitly by equations (8.44) and (8.45).

In the above family of models, we have always considered a two-dimensional cosmology at small level (or high central charge). That choice is mainly due to two obstructions that are difficult (but not necessarily impossible) to circumvent. One is associated to the difficulty of continuing from Lorentzian to Euclidean signature in the presence of electric fluxes. The other is that it is difficult to construct compact models with positive central charge deficit (negatively curved Euclidean backgrounds) in string theory, or alternatively, a compact version of linear dilaton type models. We make some further comments on this in the next section.

8.2 Liouville type models

Consider string theory cosmological backgrounds based on worldsheet CFTs of the form (see e.g. [13, 22, 14, 23 – 28]):

$$\frac{\mathrm{SL}(2,R)_{-|k|}}{\mathrm{U}(1)} \times \frac{\mathrm{SL}(2,R)_{|k|}}{\mathrm{U}(1)} \times K. \tag{8.46}$$

A nice feature of such models is that the combined central charge of the two $\mathrm{SL}(2,R)/\,\mathrm{U}(1)$ factors is independent of $|k|$, and so this can be taken to be an independent, varying parameter. Although the partition function of the euclidean cigar background is known [42], we need to deal first with the fact that for such a background the cosmological wave-function is non-normalizable due to the infinite volume of the cigar factor $\mathrm{SL}(2,R)_{|k|}/\,\mathrm{U}(1)$. To produce a normalizable wave-function we must face the problem of consistently compactifying this factor, as alluded at the end of the previous section. Moreover, it would be interesting to obtain a compactification scheme which leaves $|k|$ a free parameter. Compactifying the cigar would amount to discretizing its continuous modes keeping at the same time the unitarity and the modular invariance of the torus partition function intact.

An interesting aspect of these models is that now the sphere contribution to the string partition function is finite since the volume of the conformal Killing group cancels against the volume of the $\mathrm{SL}(2,R)_{|k|}/\,\mathrm{U}(1)$ conformal field theory factor. Nevertheless, we can see that the torus contribution dominates (at any finite string coupling) due to the volume divergence. If a consistent way of cutting off the volume of the cigar is found, we could interpret the torus contribution as a finite thermal correction to the tree-level contribution, realizing a stringy version of the computation in [5, 7, 43].

A further suggestion for developing our formalism in linear dilaton spaces, is to view the wave-function of the universe as also depending on a boundary condition in the space-like linear dilaton direction of the cigar, in order to obtain a linear dilaton holographic interpretation [44]. Part of the interpretation of the wave-function of the universe would then be as in the Hartle-Hawking picture, and part would be holographic. Finally in these models supersymmetry can be restored asymptotically in the large $|k|$ limit, making contact with linear dilaton models in null directions (see e.g. [45] for recent progress).

9. Discussion

We have outlined a framework generalizing the Hartle-Hawking no boundary proposal of the wave-function of the universe to string theory cosmological backgrounds. The class of example cosmologies considered here are described by worldsheet conformal field theories of the general form $\mathrm{SL}(2,R)_{-|k|}/\,\mathrm{U}(1) \;\; \times K$, where K is an internal, compact CFT. In order to define the analogue of the Hartle-Hawking wave-function, we had to surmount the technical hurdle of realizing that such cosmologies (like the corresponding Euclidean parafermion theories) have an almost geometrical description in terms of a compact non-singular T-fold. We then defined the wave-function of the universe via a Euclidean string field theory path integral (generalizing the no-boundary proposal). For specific examples we computed the norm of the wave-function to leading order in string perturbation theory, as a function of moduli parameters. There are many interesting similar examples to which we can generalize our analysis.

In a probabilistic interpretation, with a normalizable wave-function at hand, one can attempt to compute vacuum expectation values for particular physical quantities in various cosmological models, and analyze their properties in various regions of the moduli space. Our purpose in this paper was to provide the framework for such a discussion, which promises to be interesting. In particular, it is an open problem to identify preferred regions in the moduli space in the large class of models to which our analysis applies.

More concretely, we believe our construction points out the good use that can be made of T-folds, and generalized geometry, in stringy cosmologies (allowing to evade various no-go theorems in pure geometry). Moreover, we have been able to define a sensible calculation in a de Sitter-like compactification of string theory, after analytically continuing to the Euclidean theory. These calculations are generically hard to come by in de Sitter gravity after quantization, so any well-defined cosmological quantity, like the norm of the wave-function of stringy universes, merits scrutiny. Finally, we calculated a quantity akin to an entanglement entropy in de Sitter space, and showed that it only gets contributions starting at one loop, and we gave its microscopic origin.

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